Supplement: Case Study: Sudoku

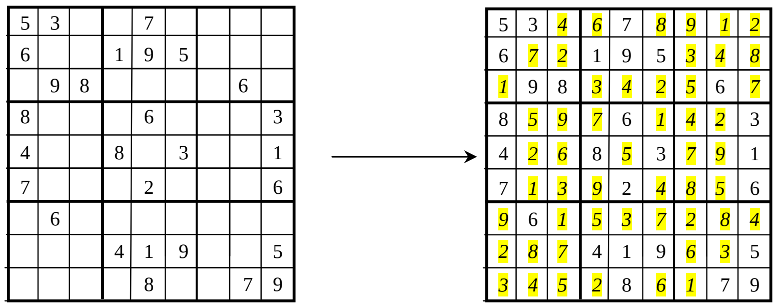
For Introduction to C++ Programming
By Y. Daniel Liang

This case study can be presented after Chapter 8, "Multidimensional Arrays."

This book teaches you how to program using a wide variety of problems with various levels of difficulty. We use simple, short, and stimulating examples to introduce programming and problem-solving techniques and use interesting and challenging examples to motivate students in programming. This section presents an interesting problem of a sort that appears in the newspaper every day. It is a number-placement puzzle, commonly known as *Sudoku*.

1 Problem Description

Sudoku is a 9 × 9 grid divided into smaller 3 × 3 boxes (also called regions or blocks), as shown in Figure 1(a). Some cells, called *fixed cells*, are populated with numbers from **1** to **9**. The objective is to fill the empty cells, also called *free cells*, with numbers **1** to **9** so that every row, every column, and every 3 × 3 box contains the numbers **1** to **9**, as shown in Figure 1(b).



5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9

5	3	*4*	*6*	7	*8*	*9*	*1*	*2*
6	*7*	*2*	1	9	5	*3*	*4*	*8*
1	9	8	*3*	*4*	*2*	*5*	6	*7*
8	*5*	*9*	*7*	6	*1*	*4*	*2*	3
4	*2*	*6*	8	*5*	3	*7*	*9*	1
7	*1*	*3*	*9*	2	*4*	*8*	*5*	6
9	6	*1*	*5*	*3*	*7*	*2*	*8*	*4*
2	*8*	*7*	4	1	9	*6*	*3*	5
3	*4*	*5*	*2*	8	*6*	*1*	7	9

(a) Input (b) Output

Figure 1
(b) is the solution to the Sudoku puzzle in (a).

For convenience, we use value **0** to indicate a free cell, as shown in Figure 2(a). The grid can be naturally represented using a two-dimensional array, as shown in Figure 2(b).

5	3	0	0	7	0	0	0	0
6	0	0	1	9	5	0	0	0
0	9	8	0	0	0	0	6	0
8	0	0	0	6	0	0	0	3
4	0	0	8	0	3	0	0	1
7	0	0	0	2	0	0	0	6
0	6	0	0	0	0	0	0	0
0	0	0	4	1	9	0	0	5
0	0	0	0	8	0	0	7	9

```
int grid[9][9] =
  {{5, 3, 0, 0, 7, 0, 0, 0, 0},
   {6, 0, 0, 1, 9, 5, 0, 0, 0},
   {0, 9, 8, 0, 0, 0, 0, 6, 0},
   {8, 0, 0, 0, 6, 0, 0, 0, 3},
   {4, 0, 0, 8, 0, 3, 0, 0, 1},
   {7, 0, 0, 0, 2, 0, 0, 0, 6},
   {0, 6, 0, 0, 0, 0, 2, 8, 0},
   {0, 0, 0, 4, 1, 9, 0, 0, 5},
   {0, 0, 0, 0, 8, 0, 0, 7, 9}
  };
```

(a) (b)

Figure 2
A grid can be represented using a two-dimensional array.

2 Problem-Solving Strategy

How do you solve this problem? An intuitive approach is to employ the following three rules:

Rule 1: Fill in free cells from the first to the last.
Rule 2: Fill in a smallest number possible.
Rule 3: If no number can fill in a free cell, backtrack.

For example, you can fill **1** into **grid[0][2]**, **2** into **grid[0][3]**, **4** into **grid[0][5]**, **8** into **grid[0][6]**, and **9** into **grid[0][7]**, as shown in Figure 3(a).

5	3	*1*	*2*	7	*4*	*8*	*9*	
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9

5	3	*1*	*2*	7	*4*	*9*	*8*	
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9

(a) (b)

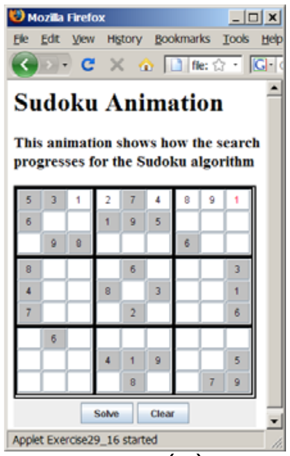
Figure 3
The program attempts to fill in free cells.

Now look at **grid[0][8]**. There is no possible value to fill in this cell. You need to backtrack to the previous free cell at **grid[0][7]** and reset its value. Since **grid[0][7]** is already **9**, no new value is possible. So you have to backtrack to its previous free cell at **grid[0][6]** and change its value to **9**. Continue to move forward to set **grid[0][7]** to **8**, as shown in Figure 3(b). Now there is still no possible value for **grid[0][8]**. Backtrack to **grid[0][7]**: no possible new value for this cell. Backtrack to **grid[0][6]**: no possible new value for this cell. Backtrack to **grid[0][5]** and change it to **6**. Now continue to move forward.

The search moves forward and backward continuously until one of the following two cases arises:

- All free cells are filled. A solution is found.
- The search is backtracked to the first free cell with no new possible value. The puzzle has no solution.

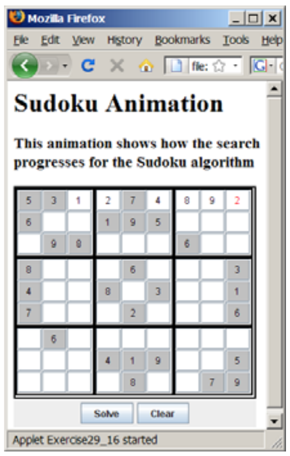
Pedagogical NOTE
Follow the link
www.cs.armstrong.edu/liang/animation/SudokuAnimation.html
to see how the search progresses. As shown in Figure 4(a), number **1** is placed in the first row and last column. This number is invalid, so the next value **2** is placed in Figure 4(b). This number is still invalid, so the next value **3** is placed in Figure 4(c). The simulation displays all the search steps.


Mozilla Firefox
File Edit View History Bookmarks Tools Help
Solve
Clear
Applet Exercise29_16 started

Sudoku Animation

This animation shows how the search progresses for the Sudoku algorithm

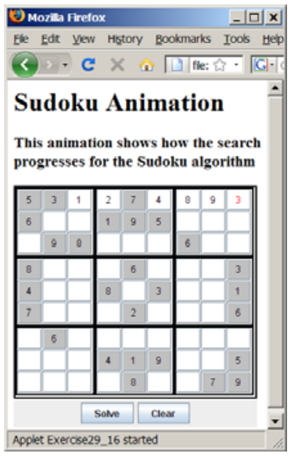
5	3	1	2	7	4	8	9	1
6			1	9	5			
	9	8				6		
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9


Mozilla Firefox
File Edit View History Bookmarks Tools Help
Solve
Clear
Applet Exercise29_16 started

Sudoku Animation

This animation shows how the search progresses for the Sudoku algorithm

5	3	1	2	7	4	8	9	2
6			1	9	5			
	9	8				6		
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9


Mozilla Firefox
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Solve
Clear
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Sudoku Animation

This animation shows how the search progresses for the Sudoku algorithm

5	3	1	2	7	4	8	9	3
6			1	9	5			
	9	8				6		
8				6				3
4			8		3			1
7				2				6
	6							
			4	1	9			5
				8			7	9

(a) (b)
(c)

Figure 4
The animation tool enables you to observe how the search works for solving a Sudoku puzzle.
******End NOTE***

3 Program Design
The program can be designed as shown in (a) and further refined with functions as in (b):

```
Read the input for a puzzle;
if (the grid is not valid)
  Report the grid not valid;
else
{
  Search for a solution;
  if (solution found)
    Display the solution;
  else
    Report no solution;
}
```

(a)

Refin $\xrightarrow{}$

```
int grid[9][9];
readAPuzzle(grid);
if (!isValid(grid))
  Report the grid not valid;
else
{
  if (search(grid))
    prindGrid(grid);
  else
    Report no solution;
}
```

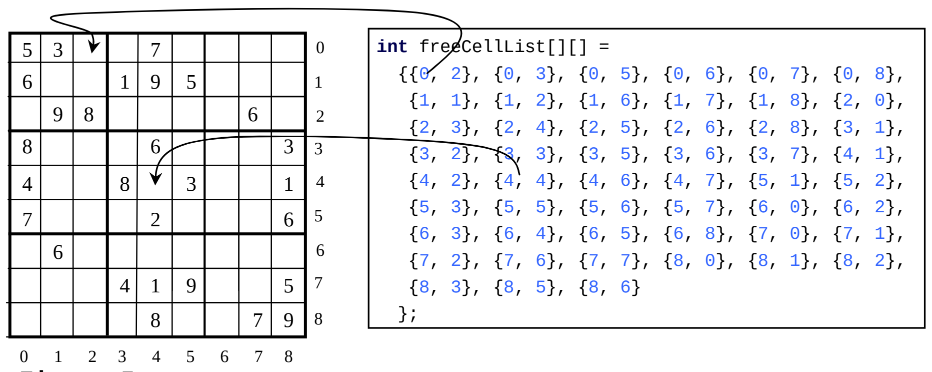
(b)

The **readAPuzzle** function reads a Sudoku puzzle from the console into **grid**. The **printGrid** function displays the contents in **grid** to the console. The **isValid** function checks whether the grid is valid. These functions are easy to implement. We now turn our attention to the **search** function.

4 Search Algorithm

To better facilitate search on free cells, the program stores free cells in a two-dimensional array, as shown in Figure 5. Each row in the array has two columns, which indicate the subscripts of the free cells in the grid. For example, {**freeCellList[0][0]**, **freeCellList[0][1]**} (i.e., {**0**, **2**}) is the subscript for the first free cell **grid[0][2]** in the grid and {**freeCellList[25][0]**, **freeCellList[25][1]**} (i.e., {**4**, **4**}) is the subscript for free cell **grid[4][4]** in the grid, as shown in Figure 5.

Comment [MSOffice1]: Not clear what 8.a is. Also, should be "as shown in" rather than "as shown".



```
int freeCellList[][] =
  {{0, 2}, {0, 3}, {0, 5}, {0, 6}, {0, 7}, {0, 8},
   {1, 1}, {1, 2}, {1, 6}, {1, 7}, {1, 8}, {2, 0},
   {2, 3}, {2, 4}, {2, 5}, {2, 6}, {2, 8}, {3, 1},
   {3, 2}, {3, 3}, {3, 5}, {3, 6}, {3, 7}, {4, 1},
   {4, 2}, {4, 4}, {4, 6}, {4, 7}, {5, 1}, {5, 2},
   {5, 3}, {5, 5}, {5, 6}, {5, 7}, {6, 0}, {6, 2},
   {6, 3}, {6, 4}, {6, 5}, {6, 8}, {7, 0}, {7, 1},
   {7, 2}, {7, 6}, {7, 7}, {8, 0}, {8, 1}, {8, 2},
   {8, 3}, {8, 5}, {8, 6}
  };
```

Figure 5
freeCellList is a two-dimensional array representation for the free cells.

The search starts from the first free cell with **k = 0**, where **k** is the index of the current free cell being considered in the free-cell list, as shown in Figure 6. It fills a valid value in the current free cell and then moves forward to consider the next. If no valid value can be found for the current free cell, the search backtracks to the preceding free cell. This process continues until all free cells are filled with valid values (a solution is found) or the search backtracks to the first free cell with no solution.

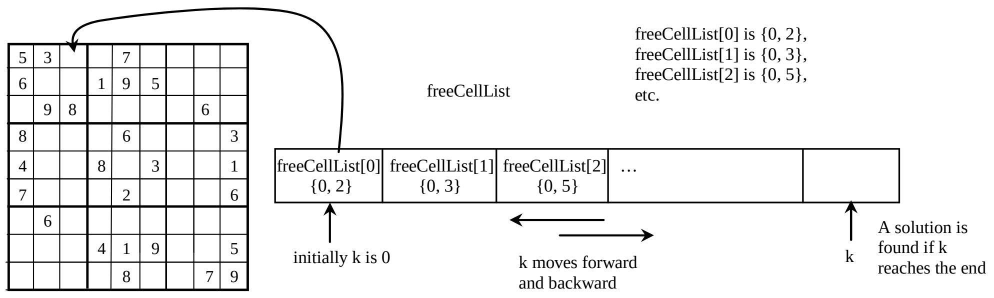

freeCellList[0] is {0, 2},
freeCellList[1] is {0, 3},
freeCellList[2] is {0, 5},
etc.
freeCellList
freeCellList[0] {0, 2}
freeCellList[1] {0, 3}
freeCellList[2] {0, 5}
…
initially k is 0
k moves forward and backward
k
A solution is found if k reaches the end

Figure 6
The search attempts to fill free cells with appropriate values.

The **search** algorithm can be described as follows:

Step 1: (Initialization) Obtain a **freeCellList** from a grid, as shown in Figure 5. Let **k** denote the index in **freeCellList** with **k** initially 0, as shown in Figure 6.

Repeatedly perform Steps 2–4 until search ends with a solution or no solution

{

```
   Step 2: Let grid[i][j] be the current free cell being considered,
           where i = freeCellList[k][0] and j = freeCellList[k][1].

   Step 3: If grid[i][j] is 0, fill it with 1.

   Step 4: Consider three cases:
      Case 1: grid[i][j] is valid. If k is the last index in
           freeCellList, a solution is found. Otherwise, search moves
           forward with k = k + 1.
      Case 2: grid[i][j] is invalid and grid[i][j] < 9. Set a new
           value for the free cell with grid[i][j] = grid[i][j] + 1.

      Case 3: grid[i][j] is invalid and grid[i][j] is 9. If k = 0,
           search ends with no solution. Otherwise backtrack with k =
           k – 1, reset i = freeCellList[k][0] and j =
           freeCellList[k][1], and continue to backtrack if grid[i][j]
           is 9. When grid[i][j] < 9, set grid[i][j] = grid[i][j] + 1.
}
```

5 Implementation

Listing 1 gives the source code for the program.

Listing 1 Sudoku.cpp

```
#include <iostream>

using namespace std;


void readAPuzzle(int grid[][9]);

bool search(int grid[][9]);

int getFreeCellList(const int grid[][9], int freeCellList[][2]);

void printGrid(const int grid[][9]);

bool isValid(int i, int j, const int grid[][9]);

bool isValid(const int grid[][9]);


int main()

{

  // Read a Sudoku puzzle

  int grid[9][9];
  readAPuzzle(grid);

  if (!isValid(grid))
    cout << "Invalid input" << endl;
  else if (search(grid))
  {
    cout << "The solution is found:" << endl;
    printGrid(grid);
  }
  else
    cout << "No solution" << endl;

  return 0;
}

// Read a Sudoku puzzle from the keyboard
void readAPuzzle(int grid[][9])
{
  cout << "Enter a Sudoku puzzle:" << endl;
  for (int i = 0; i < 9; i++)
    for (int j = 0; j < 9; j++)
      cin >> grid[i][j];
}

// Obtain a list of free cells from the puzzle
int getFreeCellList(const int grid[][9], int freeCellList[][2])
{
  // 81 is the maximum number of free cells
  int numberOfFreeCells = 0;

  for (int i = 0; i < 9; i++)
    for (int j = 0; j < 9; j++)
      if (grid[i][j] == 0)
      {
        freeCellList[numberOfFreeCells][0] = i;
        freeCellList[numberOfFreeCells][1] = j;
        numberOfFreeCells++;
      }

  return numberOfFreeCells;
}

// Display the values in the grid
void printGrid(const int grid[][9])
{
  for (int i = 0; i < 9; i++)
  {
    for (int j = 0; j < 9; j++)
      cout << grid[i][j] << " ";
    cout << endl;
  }
}

// Search for a solution
bool search(int grid[][9])
{
```

```
int freeCellList[81][2]; // Declare freeCellList
int numberOfFreeCells = getFreeCellList(grid, freeCellList);
if (numberOfFreeCells == 0)
  return true; // No free cells

int k = 0; // Start from the first free cell
while (true)
{
  int i = freeCellList[k][0];
  int j = freeCellList[k][1];
  if (grid[i][j] == 0)
    grid[i][j] = 1; // Fill the free cell with number 1

  if (isValid(i, j, grid))
  {
    if (k + 1 == numberOfFreeCells)
    { // No more free cells
      return true; // A solution is found
    }
    else
    { // Move to the next free cell
      k++;
    }
  }
  else if (grid[i][j] < 9)
  {
    // Fill the free cell with the next possible value
    grid[i][j] = grid[i][j] + 1;
    }
    else
    { // grid[i][j] is 9, backtrack
      while (grid[i][j] == 9)
      {
        if (k == 0)
        {
          return false; // No possible value
        }
        grid[i][j] = 0; // Reset to free cell
        k--; // Backtrack to the preceding free cell
        i = freeCellList[k][0];
        j = freeCellList[k][1];
      }

      // Fill the free cell with the next possible value,
      // search continues from this free cell at k
      grid[i][j] = grid[i][j] + 1;
    }
  }

  return true; // A solution is found
}

// Check whether grid[i][j] is valid in the grid
bool isValid(int i, int j, const int grid[][9])
{
  // Check whether grid[i][j] is valid at the i's row
```

```
    for (int column = 0; column < 9; column++)
      if (column != j && grid[i][column] == grid[i][j])
        return false;

    // Check whether grid[i][j] is valid at the j's column
    for (int row = 0; row < 9; row++)
      if (row != i && grid[row][j] == grid[i][j])
        return false;

    // Check whether grid[i][j] is valid in the 3-by-3 box
    for (int row = (i / 3) * 3; row < (i / 3) * 3 + 3; row++)
      for (int col = (j / 3) * 3; col < (j / 3) * 3 + 3; col++)
        if (row != i && col != j && grid[row][col] == grid[i][j])
          return false;

    return true; // The current value at grid[i][j] is valid
  }

  // Check whether the fixed cells are valid in the grid
  bool isValid(const int grid[][9])
  {
        for (int i = 0; i < 9; i++)
          for (int j = 0; j < 9; j++)
            if (grid[i][j] < 0 || grid[i][j] > 9 ||
               (grid[i][j] != 0 && !isValid(i, j, grid)))
              return false;

        return true; // The fixed cells are valid
      }
}
```

Sample Output>

```
Enter a puzzle:
0 6 0 1 0 4 0 5 0
```

```
0 0 8 3 0 5 6 0 0
2 0 0 0 0 0 0 0 1
8 0 0 4 0 7 0 0 6
0 0 6 0 0 0 3 0 0
7 0 0 9 0 1 0 0 4
5 0 0 0 0 0 0 0 2
0 0 7 2 0 6 9 0 0
0 4 0 5 0 8 0 7 0

The solution is found:
9 6 3 1 7 4 2 5 8
1 7 8 3 2 5 6 4 9
2 5 4 6 8 9 7 3 1
8 2 1 4 3 7 5 9 6
4 9 6 8 5 2 3 1 7
7 3 5 9 6 1 8 2 4
5 8 9 7 1 3 4 6 2
3 1 7 2 4 6 9 8 5
6 4 2 5 9 8 1 7 3
```

The program invokes the **readAPuzzle()** function (line 15) to read a Sudoku puzzle in a two-dimensional array **grid**. There are three possible outputs from the program:

- The input is invalid (line 17).
- A solution is found (line 19).
- No solution is found (line 24).

The **getFreeCellList** function (lines 41–56) returns a two-dimensional array storing the free-cell positions. **freeCellList[i][j]** indicates a free cell at row index **i** and column index **j**.

The **search** function invokes **getFreeCellList** to find all free cells (line 72). It then starts search from the first free cell with **k = 0** (line 77), where **k** is the position of the current free cell being considered in the free-cell list, as shown in Figure 6.

The value in a free cell starts with **1** (line 83). If the value is valid, the next cell is considered (line 93). If the value is not valid, its next value is considered (line 99). If the value is already **9**, the search is backtracked (lines 103–113). All the backtracked cells become free again and their values are reset to **0** (line 109). If the search backtracks to the free-cell list at position **k** and the current free-cell value is not **9**, increase the value by **1** (line 117) and continue the search.

The **search** function returns **true** when the search advances but no more free cells are left (line 89). A solution is found.

The search returns **false** when the search is backtracked to the first cell (line 107) and all possible values are exhausted for the cell. No solution can be found.

The **isValid(i, j, grid)** function checks whether the current value at **grid[i][j]** is valid. It checks whether **grid[i][j]** appears more than once at row **i** (lines 128–130), at column **j** (lines 133–135), and in the 3 × 3 box (lines 138–141).

How do you locate all the cells in the same box? For any **grid[i][j]**, the starting cell of the 3 × 3 box that contains it is **grid[(i / 3) * 3][(j / 3) * 3]**, as illustrated in Figure 7.

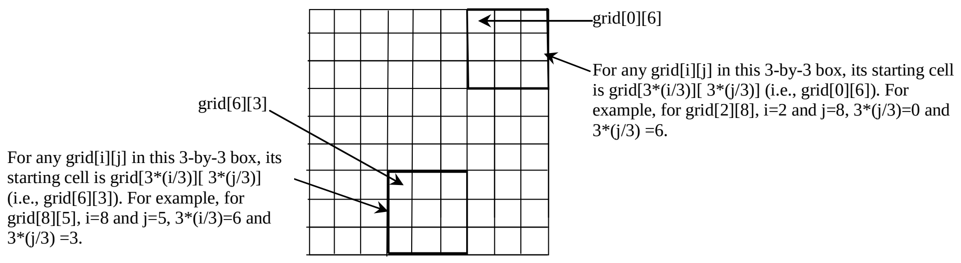

grid[0][6]
For any grid[i][j] in this 3-by-3 box, its starting cell is grid[3*(i/3)][3*(j/3)] (i.e., grid[0][6]). For example, for grid[2][8], i=2 and j=8, 3*(j/3)=0 and 3*(j/3) =6.
grid[6][3]
For any grid[i][j] in this 3-by-3 box, its starting cell is grid[3*(i/3)][3*(j/3)] (i.e., grid[6][3]). For example, for grid[8][5], i=8 and j=5, 3*(i/3)=6 and 3*(j/3) =3.

Figure 7
The location of the first cell in a 3 × 3 box determines the locations of other cells in the box.

With this observation, you can easily identify all the cells in the box. Suppose **grid[r][c]** is the starting cell of a 3 × 3 box; the cells in the box can be traversed in a nested loop as follows:

```
// Get all cells in a 3 by 3 box starting at grid[r][c]

for (int row = r; row < r + 3; row++)

  for (int col = c; col < c + 3; col++)

    // grid[row][col] is in the box
```

Note that there may be multiple solutions for an input. The program will find one such solution. You may modify the program to find all solutions in Programming Exercise 8.17.

It is cumbersome to enter 81 numbers from the keyboard. You may store the input in a file, say sudoku.txt, and compile and run the program using the following command:

```
g++ Sudoku.cpp –o Sudoku.exe
Sudoku.exe < sudoku.txt
```